DISTRIBUTIVE LATTICES, BIPARTITE GRAPHS AND ALEXANDER DUALITY

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ABSTRACT. A certain squarefree monomial ideal H_P arising from a finite partially ordered set P will be studied from viewpoints of both commutative algebra and combinatorics. First, it is proved that the defining ideal of the Rees algebra of H_P possesses a quadratic Gröbner basis. Thus in particular all powers of H_P have linear resolutions. Second, the minimal free graded resolution of H_P will be constructed explicitly and a combinatorial formula to compute the Betti numbers of H_P will be presented. Third, by using the fact that the Alexander dual of the simplicial complex Δ whose Stanley–Reisner ideal coincides with H_P is Cohen–Macaulay, all the Cohen–Macaulay bipartite graphs will be classified.

INTRODUCTION

Let P be a finite partially ordered set (*poset* for short) and write $\mathscr{J}(P)$ for the finite poset which consists of all poset ideals of P, ordered by inclusion. Here a *poset ideal* of P is a subset I of P such that if $x \in I$, $y \in P$ and $y \leq x$, then $y \in I$. In particular the empty set as well as P itself is a poset ideal of P. It follows easily that $\mathscr{J}(P)$ is a finite distributive lattice [9, p. 106]. Conversely, Birkhoff's fundamental structure theorem [9, Theorem 3.4.1] guarantees that, for any finite distributive lattice $\mathscr{L}$, there exists a unique poset P such that $\mathscr{L} = \mathscr{J}(P)$.

Let P be a finite poset with $|P| = n$, where $|P|$ is the cardinality of P, and let $S = K[\{x_p, y_p\}_{p\in P}]$ denote the polynomial ring in $2n$ variables over a field K with each $\deg x_p = \deg y_p = 1$.

We associate each poset ideal I of P with the squarefree monomial

$$u_I = (\prod_{p\in I} x_p)(\prod_{p\in P\setminus I} y_p)$$

of S of degree n. In particular $u_P = \prod_{p\in P} x_p$ and $u_\emptyset = \prod_{p\in P} y_p$.

The normal affine semigroup ring $K[\{u_I\}_{I\in\mathscr{J}(P)}]$ is studied in [6] from viewpoints of both commutative algebra and combinatorics.

In the present paper, however, we are interested in the squarefree monomial ideal

$$H_P = (\{u_I\}_{I\in\mathscr{J}(P)})$$

of S generated by all u_I with $I \in \mathscr{J}(P)$.

The outline of the present paper is as follows. First, in Section 1 we study the Rees algebra $\mathscr{R}(H_P)$ of H_P and establish our fundamental Theorem 1.1 which says that the defining ideal of $\mathscr{R}(H_P)$ possesses a reduced Gröbner basis consisting of quadratic binomials whose initial monomials are squarefree. Thus $\mathscr{R}(H_P)$ turns out to be normal and Koszul (Corollary 1.2), and all powers of H_P have linear resolutions (Corollary 1.3).

Second, in Section 2 the minimal graded free S-resolution of H_P is constructed explicitly. See Theorem 2.1. The resolution tells us how to compute the Betti numbers $\beta_i(H_P)$ of H_P in terms of the combinatorics of the distributive lattice $\mathscr{L} = \mathscr{J}(P)$. In fact, if $b_i(\mathscr{L})$ is the number of intervals $[I,J]$ of $\mathscr{L} = \mathscr{J}(P)$ which are Boolean lattices of rank i, then the ith Betti number $\beta_i(H_P)$ of H_P coincides with $b_i(\mathscr{L})$. See Corollary 2.2. (A Boolean lattice of rank i is the distributive lattice B_i which consists of all subsets of $\{1,\ldots,i\}$, ordered by inclusion.) Thus in particular for a finite distributive lattice $\mathscr{L} = \mathscr{J}(P)$, one has $\sum_{i\geq 0}(-1)^i b_i(\mathscr{L}) = 1$. See Corollary 2.3. In addition, it is shown that the ideal H_P is of height 2 and a formula to compute the multiplicity of S/H_P will be given. See Proposition 2.4 (and Corollary 2.5).

Let Δ_P denote the simplicial complex on the vertex set $\{x_p,y_p\}_{p\in P}$ such that the squarefree monomial ideal H_P coincides with the Stanley–Reisner ideal I_{Δ_P}. In Section 3 the Alexander dual $\Delta_P^\vee$ of Δ_P will be studied. Since the Stanley–Reisner ideal $H_P = I_{\Delta_P}$ has a linear resolution, it follows from [3, Theorem 3] that $\Delta_P^\vee$ is Cohen–Macaulay. It will turn out that the Stanley–Reisner ideal $I_{\Delta_P^\vee}$ of $\Delta_P^\vee$ is an edge ideal of a finite bipartite graph. Somewhat surprisingly, this simple observation enables us to classify all Cohen–Macaulay bipartite graphs. In fact, Theorem 3.4 says that a finite bipartite graph G is Cohen–Macaulay if and only if G comes from the comparability graph of a finite poset.

1. Monomial ideals arising from distributive lattices

Work with the same notation as in Introduction. Let P be a finite poset with $|P| = n$ and $S = K[\{x_p,y_p\}_{p\in P}]$ the polynomial ring in $2n$ variables over a field K with each $\deg x_p = \deg y_p = 1$. Recall that we associate each poset ideal I of P with the squarefree monomial $u_I = (\prod_{p\in I} x_p)(\prod_{p\in P\setminus I} y_p)$ of S of degree n, and introduce the ideal $H_P = (\{u_I\}_{I\in\mathscr{J}(P)})$ of S.

Let $\mathscr{R}(H_P)$ denote the Rees algebra of H_P and $\mathscr{W}_P$ the defining ideal of $\mathscr{R}(H_P)$. In other words, $\mathscr{R}(H_P)$ is the affine semigroup ring

$$\mathscr{R}(H_P) = K[\{x_p,y_p\}_{p\in P}, \{u_I t\}_{I\in\mathscr{J}(P)}] \quad (\subset K[\{x_p,y_p\}_{p\in P},t])$$

and $\mathscr{W}_P$ is the kernel of the surjective ring homomorphism $\varphi : K[\mathbf{x},\mathbf{y},\mathbf{z}] \to \mathscr{R}(H_P)$, where

$$K[\mathbf{x},\mathbf{y},\mathbf{z}] = K[\{x_p,y_p\}_{p\in P}, \{z_I\}_{I\in\mathscr{J}(P)}]$$

is the polynomial ring over K and where φ is defined by setting $\varphi(x_p) = x_p$, $\varphi(y_p) = y_p$ and $\varphi(z_I) = u_I t$.

For the convenience of our discussion, in the remainder of the present section, we will use the notation $P = \{p_1,\ldots,p_n\}$ and write x_i, y_i instead of x_{p_i}, y_{p_i}. Let $<_{lex}$ denote the lexicographic order on S induced by the ordering $x_1 > \cdots > x_n > y_1 > \cdots > y_n$ and $<^\sharp$ the reverse lexicographic order on $K[\{z_I\}_{I\in\mathscr{J}(P)}]$ induced by an ordering of the variables z_I's such that $z_I > z_J$ if $J < I$ in $\mathscr{J}(P)$. We then introduce the new monomial order $<_{lex}^\sharp$ on T by setting

$$(\prod_{i=1}^n x_i^{a_i} y_i^{b_i})(z_{I_1}\cdots z_{I_q}) <_{lex}^\sharp (\prod_{i=1}^n x_i^{a_i'} y_i^{b_i'})(z_{I_1'}\cdots z_{I_{q'}'})$$

if either

(i) $\prod_{i=1}^n x_i^{a_i} y_i^{b_i} <_{lex} \prod_{i=1}^n x_i^{a_i'} y_i^{b_i'}$

or

(ii) $\prod_{i=1}^n x_i^{a_i} y_i^{b_i} = \prod_{i=1}^n x_i^{a'_i} y_i^{b'_i}$ and $z_{I_1} \cdots z_{I_q} <^\sharp z_{I'_1} \cdots z_{I'_{q'}}$.

Theorem 1.1. *The reduced Gröbner basis $\mathscr{G}_{<^\sharp_{lex}}(\mathscr{W}_P)$ of the defining ideal $\mathscr{W}_P \subset K[\mathbf{x},\mathbf{y},\mathbf{z}]$ with respect to the monomial order $<^\sharp_{lex}$ consists of quadratic binomials whose initial monomials are squarefree.*

Proof. The reduced Gröbner basis of $\mathscr{W}_P \cap K[\{z_I\}_{I \in \mathscr{J}(P)}]$ with respect to the reverse lexicographic order $<^\sharp$ coincides with $\mathscr{G}_{<^\sharp_{lex}}(\mathscr{W}_P) \cap K[\{z_I\}_{I \in \mathscr{J}(P)}]$. It follows from [6] that $\mathscr{G}_{<^\sharp_{lex}}(\mathscr{W}_P) \cap K[\{z_I\}_{I \in \mathscr{J}(P)}]$ consists of those binomials

$$z_I z_J - z_{I \wedge J} z_{I \vee J}$$

such that I and J are incomparable in the distributive lattice $\mathscr{J}(P)$.

It is known [11, Corollary 4.4] that the reduced Gröbner basis of $\mathscr{W}_P$ consists of irreducible binomials of $K[\mathbf{x},\mathbf{y},\mathbf{z}]$. Let

$$f = (\prod_{i=1}^n x_i^{a_i} y_i^{b_i})(z_{I_1} \cdots z_{I_q}) - (\prod_{i=1}^n x_i^{a'_i} y_i^{b'_i})(z_{I'_1} \cdots z_{I'_q})$$

be an irreducible binomial of $K[\mathbf{x},\mathbf{y},\mathbf{z}]$ belonging to $\mathscr{G}_{<^\sharp_{lex}}(\mathscr{W}_P)$ with

$$(\prod_{i=1}^n x_i^{a_i} y_i^{b_i})(z_{I_1} \cdots z_{I_q})$$

its initial monomial, where $I_1 \leq \cdots \leq I_q$ and $I'_1 \leq \cdots \leq I'_q$.

Let $f \notin K[\{z_I\}_{I \in \mathscr{J}(P)}]$. Let j denote an integer for which $I'_j \not\subset I_j$. Such an integer exists. In fact, if $I'_j \subset I_j$ for all j, then each $a_i = 0$ and each $b'_i = 0$. This is impossible since $(\prod_{i=1}^n x_i^{a_i} y_i^{b_i})(z_{I_1} \cdots z_{I_q})$ is the initial monomial of f.

Let $p_i \in I'_j \setminus I_j$. Then p_i belongs to each of $I'_j, I'_{j+1}, \ldots, I'_q$, and does not belong to each of $I_1, I_2, \ldots, I_j$. Hence $a_i > 0$.

Let $p_{i_0} \in P$ with $p_{i_0} \in I'_j \setminus I_j$ such that $I_j \cup \{p_{i_0}\} \in \mathscr{J}(P)$. Thus $a_{i_0} > 0$. Let $J = I_j \cup \{p_{i_0}\}$. Then the binomial $g = x_{i_0} z_{I_j} - y_{i_0} z_J$ belongs to $\mathscr{W}_P$ with $x_{i_0} z_{I_j}$ its initial monomial. Since $x_{i_0} z_{I_j}$ divides the initial monomial of f, it follows that the initial monomial of f must coincides with $x_{i_0} z_I$, as desired. $\square$

It is well known that a homogeneous affine semigroup ring whose defining ideal has an initial ideal which is generated by squarefree (resp. quadratic) monomials is normal (resp. Koszul). See, e.g., [11, Proposition 13.15] and [4].

Corollary 1.2. *Let P be an arbitrary finite poset. Then the Rees algebra $\mathscr{R}(H_P)$ is normal and Koszul.*

On the other hand, Stefan Blum [1] proved that if the Rees algebra of an ideal is Koszul, then all powers of the ideal have linear resolutions.

Corollary 1.3. *Let P be an arbitrary finite poset. Then all powers of H_P have linear resolutions.*

2. THE FREE RESOLUTION AND BETTI NUMBERS OF H_P

Corollary 1.3 says that the monomial ideal H_P arising from a finite poset P has a linear resolution. The main purpose of the present section is to construct a minimal graded free S-resolution $\mathbb{F} = \mathbb{F}_P$ of H_P explicitly.

Let P be a finite poset with $|P| = n$ and $S = K[\{x_p, y_p\}_{p\in P}]$ the polynomial ring in $2n$ variables over a field K with each $\deg x_p = \deg y_p = 1$. Recall that, for each poset ideal I of P, we associate the squarefree monomial $u_I = (\prod_{p\in I} x_p)(\prod_{p\in P\setminus I} y_p)$ of S of degree n. Let H_P denote the ideal of S generated by all u_I with $I \in \mathscr{J}(P)$.

The maximal elements of a poset ideal I of P are called the *generators* of I. Let $M(I)$ denote the set of generators of I.

The construction of a minimal graded free S-resolution $\mathbb{F} = \mathbb{F}_P$ of H_P is achieved as follows: For all $i \geq 0$ let $\mathbb{F}_i$ denote the free S-module with basis

$$e(I,T),$$

where

$$I \in \mathscr{J}(P),\ T \subset P,\ I\cap T \subset M(I),\ |I\cap T| = i \text{ and } |I\cup T| = n+i.$$

Extending the partial order on P to a total order, we define for $i > 0$ the differential

$$\partial :\ \mathbb{F}_i \to \mathbb{F}_{i-1}$$

by

$$\partial(e(I,T)) = \sum_{p\in I\cap T} (-1)^{\sigma(I\cap T,p)}(x_p e(I\setminus\{p\},T) - y_p e(I,T\setminus\{p\})),$$

where for a subset $Q \subset P$ and $p \in Q$ we set $\sigma(Q,p) = |\{q \in Q\colon\ q < p\}|$.

With the notation introduced we have

Theorem 2.1. *The complex $\mathbb{F}$ is a graded minimal free S-resolution of H_P.*

Proof. We define an augmentation $\varepsilon\colon \mathbb{F}_0 \to H_P$ by setting

$$\varepsilon(e(I,T)) = u_I$$

for all $e(I,T) \in \mathbb{F}_0$. Note that if $e(I,T)$ is a basis element of $\mathbb{F}_0$, then $T = [n]\setminus I$, so that ε is well defined.

We first show that

$$\cdots \xrightarrow{\ \partial\ } \mathbb{F}_1 \xrightarrow{\ \partial\ } \mathbb{F}_0 \xrightarrow{\ \varepsilon\ } H_P \longrightarrow 0$$

is a complex.

Let $e(I,T) \in \mathbb{F}_1$ with $I\cap T = \{p\}$. Then

$$\begin{aligned}(\varepsilon\circ\partial)(e(I,T)) &= x_p\varepsilon(e(I\setminus\{p\},T)) - y_p\varepsilon(e(I,T\setminus\{p\})) \\ &= x_p u_{I\setminus\{p\}} - y_p u_I = 0.\end{aligned}$$

Thus $\partial\circ\varepsilon = 0$, as desired.

Next we show that $\partial \circ \partial = 0$. Let $e(I,T) \in \mathbb{F}_{i+1}$ and set $L = I \cap T$. Then

$$
\begin{aligned}
\partial \circ \partial(e(I,T)) &= \sum_{p\in L}(-1)^{\sigma(L,p)}\big(x_p\partial(e(I\setminus\{p\},T)) \\
&\quad -y_p\partial(e(I,T\setminus\{p\}))\big) \\
&= \sum_{p\in L}(-1)^{\sigma(L,p)}\big[x_p\big(\sum_{q\in L, q\neq p}(-1)^{\sigma(L\setminus\{p\},q)}\big(x_q e(I\setminus\{p,q\},T) \\
&\quad -y_q e(I\setminus\{p\},T\setminus\{q\})\big)\big) \\
&\quad -y_p\big(\sum_{q\in L, q\neq p}(-1)^{\sigma(L\setminus\{p\},q)}(x_q e(I\setminus\{q\},T\setminus\{p\}) \\
&\quad -y_q e(I,T\setminus\{p,q\})\big)\big)\big] \\
&= \sum_{p,q\in L, p\neq q}(-1)^{\sigma(L,p)+\sigma(L\setminus\{p\},q)}x_px_qe(I\setminus\{p,q\},T) \\
&\quad -\sum_{p,q\in L, p\neq q}(-1)^{\sigma(L,p)+\sigma(L\setminus\{p\},q)}x_py_qe(I\setminus\{p\},T\setminus\{q\}) \\
&\quad -\sum_{p,q\in L, p\neq q}(-1)^{\sigma(L,p)\sigma(L\setminus\{p\},q)}x_qy_pe(I\setminus\{q\},T\setminus\{p\}) \\
&\quad +\sum_{p\in L, p\neq q}(-1)^{\sigma(L,p)+\sigma(L\setminus\{p\},q)}y_py_qe(I,T\setminus\{p,q\}) \\
&= 0.
\end{aligned}
$$

The last equality holds since $(-1)^{\sigma(L,p)+\sigma(L\setminus\{p\},q)} = -(-1)^{\sigma(L,q)+\sigma(L\setminus\{q\},p)}$.

In order to prove that the augmented complex

$$\cdots \longrightarrow \mathbb{F}_1 \xrightarrow{\partial} \mathbb{F}_0 \xrightarrow{\varepsilon} H_P \longrightarrow 0$$

is exact we show:

(1) $H_0(\mathbb{F}) = H_P$,
(2) $\mathbb{F}$ is acyclic.

For the proof of (1) we note that the Taylor relations

$$r_{I,J} = x_{J\setminus I}y_{I\setminus J}e(I) - x_{I\setminus J}y_{J\setminus I}e(J), \qquad I,J \in \mathscr{J}(P)$$

generate the first syzygy module of H_P. Here we set for simplicity $e(I)$ for the basis element $e(I,P\setminus I)$ in $\mathbb{F}_0$, and denote by x_Ay_B the monomial $\prod_{p\in A}x_p\prod_{q\in B}y_q$.

Observe that

$$r_{I,J} = x_{J\setminus I}r_{I,I\cap J} - x_{I\setminus J}r_{J,I\cap J}.$$

Hence it suffices to show that $r_{I,J} \in \partial(\mathbb{F}_1)$ for all $I,J \in L$ with $J \subset I$. To this end we choose a sequence $J = I_0 \subset I_1 \subset \ldots I_{m-1} \subset I_m = I$ of poset ideals such that $I_j = I_{j-1}\cup\{p_j\}$ for $j = 1,\ldots,m$. Then

$$r_{I,J} = \sum_{j=1}^{m}\Big(\prod_{k=j+1}^{m}x_{p_k}\prod_{k=1}^{j-1}y_{p_k}\Big)r_{I_j,I_{j-1}}.$$

The assertion follows since $r_{I_j,I_{j-1}} = -\partial(e(I_j,P\setminus I_{j-1}))$ for all j.

We prove (2), that is, the acyclicity of $\mathbb{F}$ by induction on $|P|$. If $P=\{p\}$, then $H_P=(x_p,y_p)$, and $\mathbb{F}$ can be identified with the Koszul complex associated with $\{x_p,y_p\}$, and hence is acyclic.

Suppose now that $|P|>1$. Let $q\in P$ be a maximal element and let Q be the subposet $P\setminus\{q\}$. We define a map

$$\varphi\colon \mathbb{F}_Q\to\mathbb{F}_P, \qquad e_i(I,T)\mapsto e_i(I,T\cup\{q\})$$

It is clear that φ is an injective map of complexes whose induced map $H_Q=H_0(\mathbb{F}_Q)\to H_0(\mathbb{F}_P)=H_P$ is multiplication by y_q. Let $\mathbb{G}$ be the quotient complex $\mathbb{F}_P/\mathbb{F}_Q$. Since the multiplication map is injective, the short exact sequence of complexes

$$0\longrightarrow\mathbb{F}_Q\longrightarrow\mathbb{F}_P\longrightarrow\mathbb{G}\longrightarrow 0$$

induces the long exact homology sequence

$$\cdots\longrightarrow H_2(\mathbb{G})\longrightarrow H_1(\mathbb{F}_Q)\longrightarrow H_1(\mathbb{F}_P)\longrightarrow H_1(\mathbb{G})\longrightarrow 0$$

By induction hypothesis, $H_i(\mathbb{F}_Q)=0$ for $i>0$. Hence it suffices to show that $H_i(\mathbb{G})=0$ for $i>0$.

The principal order ideal (q) consists of all $p\in P$ with $p\leq p$. Let R be the subposet $P\setminus(q)$, and let $\mathbb{C}$ be the mapping cone of the complex homomorphism

$$\mathbb{F}_R\xrightarrow{-y_q}\mathbb{F}_R.$$

Then we get an exact sequence

$$0\longrightarrow\mathbb{F}_R\longrightarrow\mathbb{C}\longrightarrow\mathbb{F}_R[-1]\longrightarrow 0$$

Here $\mathbb{F}_R[-1]$ is the complex $\mathbb{F}_R$ shifted to the 'left', that is, $(\mathbb{F}_R[-1])_i=(\mathbb{F}_R)_{i-1}$ for all i.

By our induction hypothesis $\mathbb{F}_R$ is acyclic. Thus from the long exact sequence

$$\begin{array}{ccccccccc} H_1(\mathbb{C}) & \longrightarrow & H_0(\mathbb{F}_R) & \xrightarrow{-y_q} & H_0(\mathbb{F}_R) & \longrightarrow & H_0(\mathbb{C}) & \longrightarrow & 0\\ \cdots & \longrightarrow & H_2(\mathbb{C}) & \longrightarrow & H_1(\mathbb{F}_R) & \xrightarrow{-y_q} & H_1(\mathbb{F}_R) & \longrightarrow & \end{array}$$

we deduce that $H_i(\mathbb{C})=0$ for $i>1$. We also get $H_1(\mathbb{C})=0$, since $H_0(\mathbb{F}_R)=H_R$, and since multiplication by y_q is injective on H_R. Thus we see that $\mathbb{C}$ is acyclic.

We now claim that $\mathbb{C}\cong\mathbb{G}$, thereby proving that $\mathbb{G}$ is acyclic, as desired.

In order to prove this claim we first notice that $\mathbb{C}_i=(\mathbb{F}_R)_{i-1}\oplus(\mathbb{F}_R)_i$ for $i\geq 0$ (where $(\mathbb{F}_R)_{-1}=0$). Thus if $r=|R|$, then $\mathbb{C}_i$ has the basis $\mathscr{C}_i=\mathscr{B}_{i-1}\cup\mathscr{B}_i$, where

$$\begin{aligned}\mathscr{B}_i &= \{e(I,T)\colon I\in L(R),\ T\subset R,\ I\cap T\subset M(I), |I\cap T|=i,\\ &\quad |I\cup T|=r+i\}.\end{aligned}$$

On the other hand $\mathbb{G}_i$ has the basis

$$\begin{aligned}\mathscr{G}_i &= \{e(I,T)\colon I\in L(P),\ (q)\subset I,\ T\subset P,\ I\cap T\subset M(I), |I\cap T|=i,\\ &\quad |I\cup T|=n+i\}.\end{aligned}$$

Let $\psi_i:\mathbb{C}_i\to\mathbb{G}_i$ be the S-linear homomorphism with

$$\psi_i(e(I,T))=\begin{cases} e(I\cup(q),T\cup\{q\}) & \text{if } e(I,T)\in\mathscr{B}_{i-1};\\ e(I\cup(q),T) & \text{if } e(I,T)\in\mathscr{B}_i.\end{cases}$$

It is easy to see that all ψ_i are bijections and induce an isomorphism of complexes. $\square$

Suppose P is of cardinality n and P is an antichain, i.e., any two elements of P are incomparable. Then $B_n = \mathscr{J}(P)$ is called the *Boolean lattice of rank n*.

Let now $\mathscr{L}$ be an arbitrary finite distributive lattice, and let $I, J \in \mathscr{L}$ with $I \leq J$. Then the set

$$[I,J] = \{M \in \mathscr{L} \colon I \leq M \leq J\}$$

is called an *interval* in $\mathscr{L}$. The interval $[I,J]$ with the induced partial order is again a distributive lattice. Let $b_i(\mathscr{L})$ denote the number of intervals of $\mathscr{L}$ which are isomorphic to Boolean lattices of rank i. In particular, $b_0(\mathscr{L}) = |\mathscr{L}|$. These numbers have an algebraic interpretation.

Recall that for a graded S-module M,

$$\beta_i(M) = \dim_K \operatorname{Tor}_i^S(M,K)$$

is called the *ith Betti-number of M*. If $\mathbb{F}$ is a graded minimal free resolution of M, then $\beta_i(M)$ is nothing but the rank of $\mathbb{F}_i$.

Corollary 2.2. *Let P be a finite poset, $\mathscr{L} = \mathscr{J}(P)$ the distributive lattice and H_P the squarefree monomial ideal arising from P. Then*

(a) *$b_i(\mathscr{L}) = \beta_i(H_P)$ for all i;*

(b) *the following three numbers are equal:*

  (i) *the projective dimension of H_P;*

  (ii) *the maximum of the ranks of Boolean lattices which are isomorphic to an interval of $\mathscr{L}$;*

  (iii) *the Sperner number of P, i.e., the maximum of the cardinalities of antichains of P.*

Proof. (a) For each $i \geq 0$, let $\mathscr{S}_i$ be the set of pairs (I,S), where $I \in \mathscr{L}$, $S \subset M(I)$ and $|S| = i$, and let $\mathscr{B}_i$ be the set of basis elements $e(I,T)$ of $\mathbb{F}_i$. Then

$$\mathscr{B}_i \longrightarrow \mathscr{S}_i, \quad e(I,T) \mapsto (I, I \cap T)$$

establishes a bijection between these two sets.

Since for each $(I,S) \in \mathscr{S}_i$, the elements in S are pairwise incomparable it is clear that $[I \setminus S, I]$ is isomorphic to a Boolean lattice of rank i.

Conversely, suppose $[J,I]$ is isomorphic to a Boolean lattice of rank i. Then $S = I \setminus J$ is of a set of cardinality i, and $J \cup T \in \mathscr{L}$ for all subsets $T \subset S$.

Suppose that $S \not\subset M(I)$. Then there exists, $q \in S$ and $p \in I$ such that $p > q$. If $p \in J$, then $q \in J$, a contradiction. Thus $p \in S$, and hence $(J,p) \in \mathscr{L}$. This is again a contradiction, because it would imply that $q \in (J,p)$. Hence we have shown that $(I,S) \in \mathscr{S}_i$.

It follows that the assignment $e(I,T) \mapsto [I \setminus (I \cap T), I]$ establishes a bijection between the basis of $\mathbb{F}_i$ and the intervals of $[J,I]$ in $\mathscr{L}$ which are isomorphic to Boolean lattices.

(b) is an immediate consequence of (a) and its proof. □

Corollary 2.3. *Let $\mathscr{L}$ be a finite distributive lattice. Then*

$$\sum_{i \geq 0} (-1)^{i+1} b_i(\mathscr{L}) = 1.$$

Let Δ_P be the simplicial complex attached to the squarefree monomial ideal H_P. In the next section we will see (Lemma 3.1) that the Stanley–Reisner ideal attached to the Alexander dual $\Delta_P^\vee$ is generated by the monomials $x_p y_q$ such that $p \leq q$. Hence for the Stanley–Reisner ideal of Δ_P we have

$$I_{\Delta_P} = \bigcap_{p,q \in P,\ p \leq q} (x_p, x_q).$$

In particular we get

Proposition 2.4. *Let P be a finite poset. Then the squarefree monomial ideal H_P is of height 2, and the multiplicity of S/H_P is given by*

$$e(S/H_P) = |\{(p,q)\colon\ p,q \in P,\ p \leq q\}|.$$

Let $I \subset S$ be an arbitrary graded ideal with graded minimal free resolution

$$0 \longrightarrow \bigoplus_{j=1}^{\beta_s} S(-a_{sj}) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{\beta_1} S(-a_{1j}) \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

Suppose the height of I equals h. Then by a formula of Peskine and Szpiro [8] one has

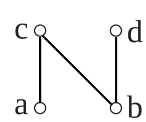
$$e(S/I) = \frac{(-1)^h}{h!} \sum_{i=1}^{s} (-1)^i \sum_{j=1}^{\beta_i} a_{ij}^h.$$

Applying this formula in our situation and using Corollary 2.2 and Proposition 2.4 we get

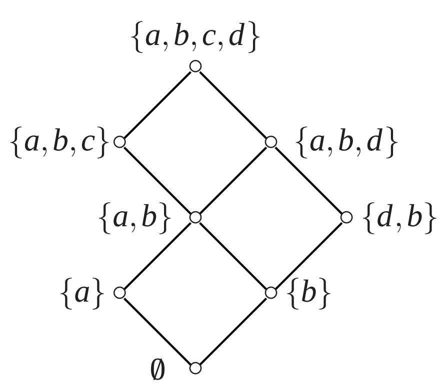
Corollary 2.5. *Let P be a finite poset with $|P| = n$, and let $\mathscr{L} = \mathscr{J}(P)$ be the distributive lattice. Then*

$$|\{(p,q)\colon\ p,q \in P,\ p \leq q\}| = \frac{1}{2} \sum_{i \geq 0} (-1)^{i+1} b_i(\mathscr{L})(n+i)^2.$$

We close this section with an example. Let P be the poset with Hasse diagram



The distributive lattice $\mathscr{L} = \mathscr{J}(P)$ has the Hasse diagram



Thus $H_P = (uvwx, avwx, buwx, abwx, bduw, abcx, abdw, abcd)$. Here we use for convenience the indeterminates a,b,c,d,u,v,w,x instead of x_p and y_p. The free resolution of H_P is given by

$$0 \longrightarrow S^3(-6) \longrightarrow S^{10}(-5) \longrightarrow S^8(-4) \longrightarrow H_P \longrightarrow 0.$$

We see from the Hasse diagram that the ith Betti number of H_P coincides with number of intervals of $\mathscr{L}$ which are isomorphic to Boolean lattices of rank i. The number of pairs (p,q) in the poset P with $p \leq q$ is equal to 7, and this is also the number we get from Corollary 2.5, namely $(1/2)(-8 \cdot 16 + 10 \cdot 25 - 3 \cdot 36) = 7$.

3. Alexander duality and Cohen–Macaulay bipartite graphs

We refer the reader to, e.g., [10], [2] and [7] for fundamental information about Stanley–Reisner rings.

Let $P = \{p_1, \ldots, p_n\}$ be a finite poset and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ the polynomial ring in $2n$ variables over a field K with each $\deg x_i = \deg y_i = 1$. We will use the notation x_i, y_i instead of x_{p_i}, y_{p_i}, and set $V_n = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$.

Recall that H_P is the ideal of S which is generated by those squarefree monomials $u_I = (\prod_{p_i \in I} x_i)(\prod_{p_i \in P \setminus I} y_i)$ with $I \in \mathscr{J}(P)$. It then follows that there is a unique simplicial complex Δ_P on V_n such that the *Stanley–Reisner ideal* I_{Δ_P} coincides with H_P. We study the *Alexander dual* $\Delta_P^\vee$ of Δ_P, which is the simplicial complex

$$\Delta_P^\vee = \{V_n \setminus F \,:\, F \not\in \Delta_P\}$$

on V_n.

Lemma 3.1. *The Stanley–Reisner ideal of $\Delta_P^\vee$ is generated by those squarefree quadratic monomials $x_i y_j$ such that $p_i \leq p_j$ in P.*

Proof. Let $w = x_1 \cdots x_n y_1 \cdots y_n$. If u is a squarefree monomial of S, then we write $\mathrm{supp}(u)$ for the support of u, i.e., $\mathrm{supp}(u) = \{x_i \,:\, x_i \text{ divides } u\} \cup \{y_j \,:\, y_j \text{ divides } u\}$. Now since $\{\mathrm{supp}(u_I) : I \in \mathscr{J}(P)\}$ is the set of minimal nonfaces of Δ_P, it follows that $\{\mathrm{supp}(w/u_I) : I \in \mathscr{J}(P)\}$ is the set of facets (maximal faces) of $\Delta_P^\vee$. Our work is to find the minimal nonfaces of $\Delta_P^\vee$. Since $\mathrm{supp}(w/u_\emptyset) = x_1 \cdots x_n$ and $\mathrm{supp}(w/u_P) = y_1 \cdots y_n$, both $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are faces of $\Delta_P^\vee$. Let $F \subset V_n$ be a nonfaces of $\Delta_P^\vee$. Let $F_x = F \cap \{x_1, \ldots, x_n\}$ and $F_y = \{x_j \,:\, y_j \in F\}$. Then $F_x \neq \emptyset$ and $F_y \neq \emptyset$. Since $\{x_i, y_i\}$ is a minimal nonface of $\Delta_P^\vee$, we will assume that $F_x \cap F_y = \emptyset$. Since F is a nonface, there exists *no* poset ideal I of P with $F_x \cap \{x_i \,:\, p_i \in I\} = \emptyset$ and $F_y \subset \{x_i \,:\, p_i \in I\}$. Hence there are $x_i \in F_x$ and $x_j \in F_y$ such that $p_i < p_j$. Thus $\{x_i, y_j\}$ is a nonface of $\Delta_P^\vee$. Hence the set of minimal nonfaces of $\Delta_P^\vee$ consists of those 2-element subsets $\{x_i, y_j\}$ of V_n such that $p_i \leq p_j$ in P, as required. $\square$

Let G be a finite graph on the vertex set $[N] = \{1, \ldots, N\}$ with no loops and no multiple edges. We will assume that G possesses no isolated vertex, i.e., for each vertex i there is an edge e of G with $i \in e$. A *vertex cover* of G is a subset $C \subset [N]$ such that, for each edge $\{i, j\}$ of G, one has either $i \in C$ or $j \in C$. Such a vertex cover C is called *minimal* if no subset $C' \subset C$ with $C' \neq C$ is a vertex cover of G. We say that a finite graph G is *unmixed* if all minimal vertex covers of G have the same cardinality.

Let $K[\mathbf{z}] = K[z_1, \ldots, z_N]$ denote the polynomial ring in N variables over a field K. The *edge ideal* of G is the ideal $I(G)$ of $K[\mathbf{z}]$ generated by those squarefree quadratic monomials $z_i z_j$ such that $\{i, j\}$ is an edge of G. A finite graph G on $[N]$ is called *Cohen–Macaulay* over K if the quotient ring $K[\mathbf{z}]/I(G)$ is Cohen–Macaulay. Every Cohen–Macaulay graph is unmixed ([12, Proposition 6.1.21]).

A finite graph G on $[N]$ is *bipartite* if there is a partition $[N] = W \cup W'$ such that each edge of G is of the form $\{i, j\}$ with $i \in W$ and $j \in W'$. A basic fact on the graph theory says that a finite graph G is bipartite if and only if G possesses no cycle of odd length. A *tree* is a connected graph with no cycle. A tree is Cohen–Macaulay if and only if it is unmixed ([12, Corollary 6.3.5]).

Given a finite poset $P = \{p_1, \ldots, p_n\}$, we write $G(P)$ for the bipartite graph on the vertex set $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ whose edges are those $\{x_i, y_j\}$ such that $p_i \leq p_j$ in P. Lemma 3.1 says that the Stanley–Reisner ideal of $\Delta_P^\vee$ is equal to the edge ideal of $G(P)$. Since the Stanley–Reisner ideal $H_P = I_{\Delta_P}$ has a linear resolution, it follows from [3, Theorem 3] that $\Delta_P^\vee$ is Cohen–Macaulay. Then [12, Theorem 6.4.7] says that $\Delta_P^\vee$ is shellable. Hence I_{Δ_P} has linear quotients (e.g., [5]).

Corollary 3.2. *The ideal H_P has linear quotients.*

We now turn to the problem of classifying the Cohen–Macaulay bipartite graphs by using the Alexander dual $\Delta_P^\vee$.

Let G be a finite bipartite graph on the vertex set $W \cup W'$ with $W = \{i_1, \ldots, i_s\}$ and $W' = \{j_1, \ldots, j_t\}$, where $s \leq t$. For each subset U of W, we write $N(U)$ for the set of those vertices $j \in W'$ for which there is a vertex $i \in U$ such that $\{i, j\}$ is an edge of G. The well-known "marriage theorem" in graph theory says that if $|U| \leq |N(U)|$ for all subsets U of W, then there is a subset $W'' = \{j_{\ell_1}, \ldots, j_{\ell_s}\} \subset W'$ with $|W''| = s$ such that $\{i_k, j_{\ell_k}\}$ is an edge of G for $k = 1, 2, \ldots, s$.

Let G be a finite bipartite graph on the vertex set $W \cup W'$ and suppose that G is unmixed. Since each of W and W' is a minimal vertex cover, one has $|W| = |W'|$. Let $W = \{x_1, \ldots, x_n\}$ and $W' = \{y_1, \ldots, y_n\}$. Since $(W \setminus U) \cup N(U)$ is a vertex cover of G for all subsets U of W and since G is unmixed, it follows that $|U| \leq |N(U)|$ for all subsets U of W. Thus the marriage theorem enables us to assume that G satisfies the condition as follows: $(\sharp)$ $\{x_i, y_i\}$ is an edge of G for all $1 \leq i \leq n$.

Lemma 3.3. *Work with the same notation as above and, furthermore, suppose that G is a Cohen–Macaulay graph. Then, after a suitable change of the labeling of variables $y_1, \ldots, y_n$, the edge set of G satisfies the condition $(\sharp)$ together with the condition as follows: $(\sharp\sharp)$ if $\{x_i, y_j\}$ is an edge of G, then $i \leq j$.*

Proof. Let Δ be the Cohen–Macaulay complex on the vertex set $W \cup W'$ whose Stanley–Reisner ideal I_Δ coincides with $I(G)$. Recall that every Cohen–Macaulay complex is strongly connected and that all links of a Cohen–Macaulay complex are again Cohen–Macaulay. Since both W and W' are facets of Δ, it follows (say, by induction on n) that, after a suitable change of the labeling of variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, the subset $F_i = \{y_1, \ldots, y_i, x_{i+1}, \ldots, x_n\}$ is a facet of Δ for each $0 \leq i \leq n$, where $F_0 = W$ and $F_n = W'$. In particular $\{x_i, y_j\}$ cannot be an edge of G if $j < i$. In other words, the edge set of G satisfies the conditions $(\sharp)$ and $(\sharp\sharp)$, as required. $\square$

Theorem 3.4. *Let G be a finite bipartite graph on the vertex set $W \cup W'$, where $W = \{x_1, \ldots, x_n\}$ and $W' = \{y_1, \ldots, y_n\}$, and suppose that the edge set of G satisfies the conditions $(\sharp)$ and $(\sharp\sharp)$. Then G is a Cohen–Macaulay graph if and only if the following condition $(\sharp\sharp\sharp)$ is satisfied:*

$(\sharp\sharp\sharp)$ *If $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are edges of G with $i < j < k$, then $\{x_i, y_k\}$ is an edge of G.*

Proof. **("Only if")** Let G be a Cohen–Macaulay graph satisfying $(\sharp)$ and $(\sharp\sharp)$ and Δ the Cohen–Macaulay complex on the vertex set $W \cup W'$ whose Stanley–Reisner ideal coincides with $I(G)$. Let $\{x_i, y_j\}$ and $\{x_j, y_k\}$ be edges of G with $i < j < k$ and suppose that $\{x_i, y_k\}$ is not an edge of G. Since every Cohen–Macaulay complex is pure and since $\{x_i, y_k\}$ is a face of Δ, it follows that there is an n-element subset $F \subset W \cup W'$ of G with $\{x_i, y_k\} \subset F$ such that F is independent in G, i.e., no 2-element subset of F is an edge of G. One has $y_j \notin F$ and $x_j \notin F$ since $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are edges of G. Since $\{x_\ell, y_\ell\}$ is an edge of G for each $1 \leq \ell \leq n$, the independent subset F can contain both x_i and y_i for no $1 \leq i \leq n$. Thus to find such an n-element independent set F is impossible.

("If") Now, suppose that a finite bipartite graph G on the vertex set $W \cup W'$ satisfies the conditions $(\sharp)$, $(\sharp\sharp)$ together with $(\sharp\sharp\sharp)$. Let $\leq$ denote the binary relation on $P = \{p_1, \ldots, p_n\}$ defined by setting $p_i \leq p_j$ if $\{x_i, y_j\}$ is an edge of G. By $(\sharp)$ one has $p_i \leq p_i$ for each $1 \leq i \leq n$. By $(\sharp\sharp)$ if $p_i \leq p_j$ and $p_j \leq p_i$, then $p_i = p_j$. By $(\sharp\sharp\sharp)$ if $p_i \leq p_j$ and $p_j \leq p_k$, then $p_i \leq p_k$. Thus $\leq$ is a partial order on P. Lemma 3.1 then guarantees that $G = G(P)$. Hence G is Cohen–Macaulay, as desired. $\square$

Corollary 3.5. *Let G be a finite bipartite graph and Δ the simplicial complex whose Stanley–Reisner ring coincides with $I(G)$. Then G is Cohen–Macaulay if and only if Δ is pure and strongly connected.*

Work with the same situation as in the "if" part of the proof of Theorem 3.4. Let $\mathrm{com}(P)$ denote the *comparability graph* of P, i.e., $\mathrm{com}(P)$ is the finite graph on $\{p_1, \ldots, p_n\}$ whose edges are those $\{p_i, p_j\}$ with $i \neq j$ such that p_i and p_j are comparable in P. It then follows from [12, pp. 184 – 185] that the Cohen–Macaulay type of the Cohen–Macaulay ring $S/I(G)$, where $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, is the number of maximal independent subsets of $\mathrm{com}(P)$, i.e., the number of maximal antichains of P. Hence G is Gorenstein, i.e., $S/I(G)$ is a Gorenstein ring, if and only if P is an antichain.

Corollary 3.6. *A Cohen-Macaulay bipartite graph G is Gorenstein if and only if G is the disjoint union of edges.*

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